

Problem 2.35

[Difficulty: 4]

2.35 A flow is described by velocity field $\vec{V} = ay\hat{i} + b\hat{j}$, where $a = 0.2 \text{ s}^{-1}$ and $b = 0.4 \text{ m/s}^2$. At $t = 2 \text{ s}$, what are the coordinates of the particle that passed through point (1, 2) at $t = 0$? At $t = 3 \text{ s}$, what are the coordinates of the particle that passed through point (1, 2) at $t = 2 \text{ s}$? Plot the pathline and streakline through point (1, 2), and plot the streamlines through the same point at the instants $t = 0, 1, 2$, and 3 s .

Given: Velocity field

Find: Coordinates of particle at $t = 2 \text{ s}$ that was at (1,2) at $t = 0$; coordinates of particle at $t = 3 \text{ s}$ that was at (1,2) at $t = 2 \text{ s}$; plot pathline and streakline through point (1,2) and compare with streamlines through same point at $t = 0, 1$ and 2 s

Solution

Governing equations: For pathlines $u_p = \frac{dx}{dt}$ $v_p = \frac{dy}{dt}$ For streamlines $\frac{v}{u} = \frac{dy}{dx}$

Following the discussion leading up to Eq. 2.10, we first find equations for the pathlines in form

$$x_p(t) = x(t, x_0, y_0, t_0) \quad \text{and} \quad y_p(t) = y(t, x_0, y_0, t_0)$$

$$x_{st}(t_0) = x(t, x_0, y_0, t_0) \quad \text{and} \quad y_{st}(t_0) = y(t, x_0, y_0, t_0)$$

which gives the streakline at t , where x_0, y_0 is the point at which dye is released (t_0 is varied from 0 to t)

Assumption: 2D flow

Given data $a = 0.2 \frac{1}{s} \quad b = 0.4 \frac{m}{s^2}$

Hence for pathlines $u_p = \frac{dx}{dt} = a \cdot y \quad v_p = \frac{dy}{dt} = b \cdot t$

Hence $dx = a \cdot y \cdot dt \quad dy = b \cdot t \cdot dt \quad y - y_0 = \frac{b}{2} \cdot (t^2 - t_0^2)$

For x $dx = \left[a \cdot y_0 + a \cdot \frac{b}{2} \cdot (t^2 - t_0^2) \right] \cdot dt$

Integrating $x - x_0 = a \cdot y_0 \cdot (t - t_0) + a \cdot \frac{b}{2} \cdot \left[\frac{t^3}{3} - \frac{t_0^3}{3} - t_0^2 \cdot (t - t_0) \right]$

The pathlines are $x(t) = x_0 + a \cdot y_0 \cdot (t - t_0) + a \cdot \frac{b}{2} \cdot \left[\frac{t^3}{3} - \frac{t_0^3}{3} - t_0^2 \cdot (t - t_0) \right] \quad y(t) = y_0 + \frac{b}{2} \cdot (t^2 - t_0^2)$

These give the position (x,y) at any time t of a particle that was at (x_0, y_0) at time t_0

Note that streaklines are obtained using the logic of the Governing equations, above

The streaklines are

$$x(t_0) = x_0 + a \cdot y_0 \cdot (t - t_0) + a \cdot \frac{b}{2} \cdot \left[\frac{t^3}{3} - \frac{t_0^3}{3} - t_0^2 \cdot (t - t_0) \right] \quad y(t_0) = y_0 + \frac{b}{2} \cdot (t^2 - t_0^2)$$

These gives the streakline at t , where x_0, y_0 is the point at which dye is released (t_0 is varied from 0 to t)

For a particle that was at $x_0 = 1 \text{ m}$, $y_0 = 2 \text{ m}$ at $t_0 = 0 \text{ s}$, at time $t = 2 \text{ s}$ we find the position is (from pathline equations)

$$x = x_0 + a \cdot y_0 \cdot (t - t_0) + a \cdot \frac{b}{2} \cdot \left[\frac{t^3}{3} - \frac{t_0^3}{3} - t_0^2 \cdot (t - t_0) \right] = 1.9 \text{ m} \quad y = y_0 + \frac{b}{2} \cdot (t^2 - t_0^2) = 2.8 \text{ m}$$

For a particle that was at $x_0 = 1 \text{ m}$, $y_0 = 2 \text{ m}$ at $t_0 = 2 \text{ s}$, at time $t = 3 \text{ s}$ we find the position is

$$x = x_0 + a \cdot y_0 \cdot (t - t_0) + a \cdot \frac{b}{2} \cdot \left[\frac{t^3}{3} - \frac{t_0^3}{3} - t_0^2 \cdot (t - t_0) \right] = 1.4 \text{ m} \quad y = y_0 + \frac{b}{2} \cdot (t^2 - t_0^2) = 3.0 \text{ m}$$

For streamlines

$$\frac{v}{u} = \frac{dy}{dx} = \frac{b \cdot t}{a \cdot y}$$

So, separating variables

$$y \cdot dy = \frac{b}{a} \cdot t \cdot dx \quad \text{where we treat } t \text{ as a constant}$$

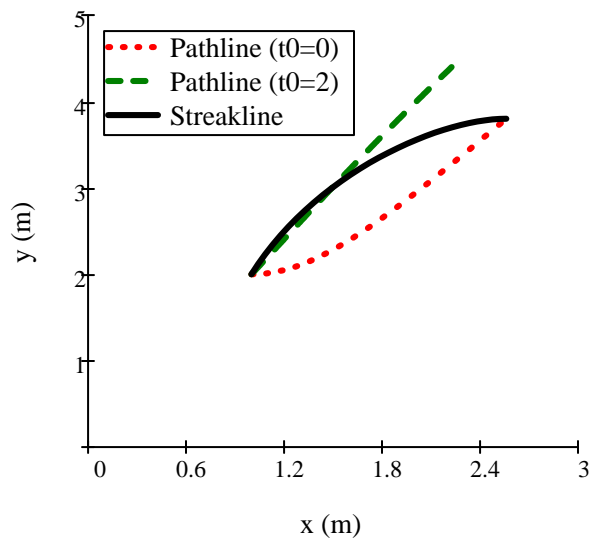
Integrating

$$\frac{y^2 - y_0^2}{2} = \frac{b \cdot t}{a} \cdot (x - x_0) \quad \text{and we have} \quad x_0 = 1 \text{ m} \quad y_0 = 2 \text{ m}$$

The streamlines are then

$$y = \sqrt{y_0^2 + \frac{2 \cdot b \cdot t}{a} \cdot (x - x_0)} = \sqrt{4 \cdot t \cdot (x - 1) + 4}$$

Pathline Plots



Streamline Plots

